Joint spectrum and spectral radius of random products of matrices

05.05.2020

1) Joint spectrum 2) Random products 3) Spectral radius

$G = SL_2(\mathbb{R})$ or $SL_3(\mathbb{R})$

$S \subseteq G$ S → finite, → infinite, bounded.

$S^n = \{ g_1 \dots g_n \mid g_i \in S \} \subseteq G$

Tools to see: $G = KA^+K$ $G = SL_d(\mathbb{R})$, $K = SO_d(\mathbb{R})$

$A^+ = \left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_d \end{pmatrix} \;\; \begin{matrix} \prod \alpha_i = 1 \\ \alpha_1 \geq \dots \geq \alpha_d > 0 \end{matrix} \right\}$

→ multi-norm Cartan projection $\underline{K(g) = (\log \alpha_1, \dots, \log \alpha_d) \subseteq \mathbb{R}^{d-1}}$

α_i's → singular values

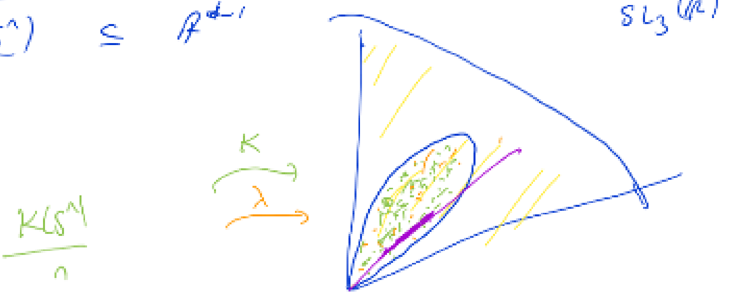
$\alpha_1 = \|g\|$

→ Jordan projection / moduli of eigenvalues

$\lambda(g) = (\log \lambda_1, \dots, \log \lambda_d) \subseteq \mathbb{R}^{d-1}$ where $\lambda_1 \geq \dots \geq \lambda_d > 0$ are moduli of eigenvalues of g.

S^n $\quad \frac{K(S^n)}{n}, \frac{\lambda(S^n)}{n} \subseteq \mathbb{R}^{d-1}$

$SL_3(\mathbb{R})$



Spectral radius formula

$\|g^n\|^{1/n} \to \lambda_1(g)$

$\frac{1}{n} K(g^n) \to \lambda(g)$

Q: 1) Do these sequences converge for the H-metric?

2) Same limit? How do the limits look like?

Thm: (Breuillard – S, '18)

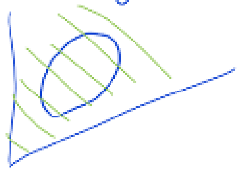
Suppose S generates a Zariski-dense semigroup $\bigcup_{n \geq 1} S^n$. Then,

① $\frac{K(S^n)}{n}$ and $\frac{\lambda(S^n)}{n}$ converge to a compact set, Joint spectrum $J(S)$ of S.

② $J(S)$ is convex and non-empty interior.

Rk: 1) Without Z-density non-empty interior is not true.
⟹ convexity fails.

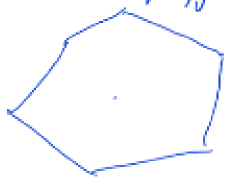
① Continuity: In general, it's not continuous.
$S \mapsto J(S)$



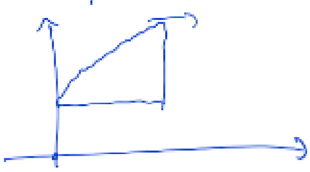
if $J(S)$ does not touch the walls (projective uniform hyperbolicity / domination) ⟹ S is a continuity point. (Bochi–Morris)

Rk: The joint spectrum is a segment in $SL_2(\mathbb{R})$
$\rho_{sub}(S) \longrightarrow \rho(S)$ joint spectral radius, $\boxed{\exp \rho(S) = \lim_{n\to\infty} \sup_{g \in S^n} \|g\|^{1/n}}$

② Realizability: $\forall$ K = convex body in $\mathfrak{a}^+$, $\exists$ S Zariski dense s.t. $K = J(S)$. Moreover, if K is a polygon ⟹ $\exists$ finite set S Z-dense, $K = J(S)$.



③ $\exists$ $S \subseteq SL_2 \times SL_2$ or GL_2 (Bochi)
s.t. $J(S)$ is not polygonal, in fact its boundary has a dense set of non-differentiable points.



$(1,1)$
↓

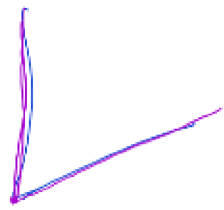
2) Random products of matrices

G, let μ be a proba on G. $\mu = \frac{1}{2}\delta_{(\,)} + \frac{1}{2}\delta_{(\,)}$
let X_1, X_2 be iid dist $\sim \mu$.
$L_n = X_n \cdots X_1$. $\log \|L_n\|$ = first coeff of ... | $GL_2(\mathbb{R}) \simeq \mathbb{R}^*$
↓

LLN, CLT, Large deviations. | classical setting.

Furstenberg – Kesten 60' : Under first moment assumption

$$\underset{\text{multinorm}}{\frac{1}{n} K(L_n)} \xrightarrow{a.s.} \vec{\lambda}_\mu = (\underset{\text{top Lyapunov exp.}}{\lambda_1}, \lambda_2, \dots, \lambda_d) \in \mathfrak{a}^+$$



Furstenberg 60' : $\underbrace{\overbrace{\text{Supp}\,\mu}^{=S} \text{ is Zariski-dense}}$ $\Rightarrow$

$\vec{\lambda}_\mu \neq 0 \quad \Rightarrow \quad \lambda_1 > 0$

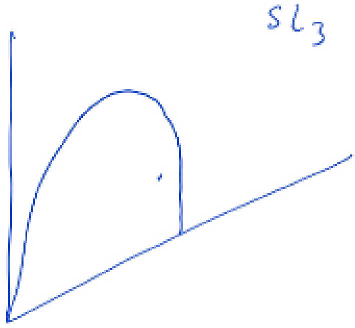
Guivarc'h–Raugi + Goldsheid + Margulis 80' : Supp μ is Z-dense

$\Rightarrow \quad \lambda_\mu \notin$ walls. $\Rightarrow \quad \lambda_1 > \lambda_2 > \dots > \lambda_d$ (simplicity of Lyapunov spectrum)

Thm: (Breuillard – S 18') $\quad S = \text{supp}(\mu)$

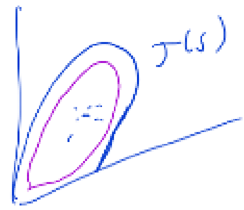
$$\frac{1}{n} K(\underbrace{L_n}_{= \underbrace{X_n \cdots X_1}_{\in S^n}}) \xrightarrow{a.s.} \vec{\lambda}_\mu \in \mathring{J}(S)$$

$\frac{1}{n} K(S^n) \to J(S)$



Consider : Fix S, start varying μ

Ex: $S = \left\{ \underbrace{\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}}_{a}, \underbrace{\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}}_{b} \right\} \subseteq SL_2(\mathbb{R})$



$\rho(S) \overset{\text{def}}{=} \lim_{n \to \infty} \sup_{g \in S^n} \|g\|^{1/n}$

$\mu = p\,\delta_a + (1-p)\,\delta_b$

$\rightarrow$ Bernoulli measures are not sufficient to fill $J(S)$ with their Lyapunov vector.

Fact: It's a general fact from subadditive ergodic theory that $\exists$ always a shift-inv. ergodic prob. μ on $\boxed{S^{\mathbb{Z}}}$ s.t. $\lambda_1(\mu) = \rho(S)$. (Morris '10)

Thm: (Breuillard - S) S = Z-dense. $\forall x \in \mathring{J}(S)$ $\exists \mu$ shift-inv, ergodic prob. on $S^{\mathbb{N}}$ s.t. $\vec{\lambda}_\mu = x$.

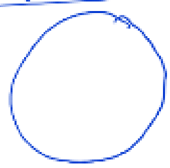
$SL_2(\mathbb{R})$. Structure of measures hitting the boundary.

Thm: (Bochi - Rams '13)

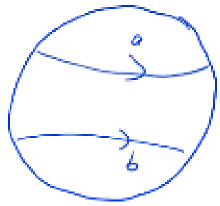
$SL_2(\mathbb{R})$ S finite set which plays ping-pong.

$\Rightarrow$ $\forall \mu$ prob on $S^{\mathbb{N}}$ s.t.

$\lambda_1(\mu) = \rho(S)$ $\Rightarrow$ μ has 0-entropy.



|| Jenkinson - Pollicott / Breuillard - S:



$S = \{a, b\} \subseteq SL_2(\mathbb{R})$

↓ hyperbolic playing ping-pong

$\Rightarrow$ $\exists!$ μ on $S^{\mathbb{N}}$ s.t. $\rho(S) = \lambda_1(\mu)$

$\Rightarrow$ μ is a Sturmian prob. measure. $\rightarrow \alpha \in [0,1)$

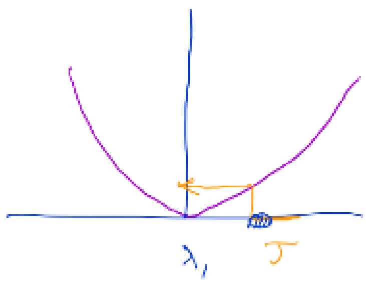
↓ exhibit lowest complexity. Morse-Thue

3). Large deviations; if $\mathring{J}(\mu) \ni \lambda$

$$\mathbb{P}\Big(\overbrace{\tfrac{1}{n} \log \|L_n\|}^{\to \lambda_1} \in J \Big) \to 0$$

$$\leq$$

$$\| \quad e^{-n(\min_{\alpha \in J} I(\alpha))}$$



1) || Existence of LDP (large dev. principle) = existence of such a function.

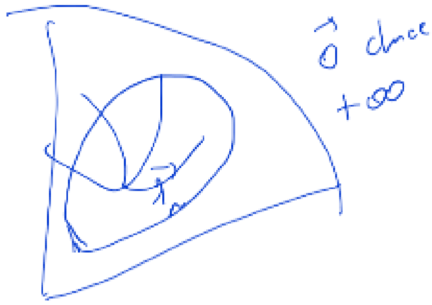
2) Exponential decay.

Thm (S '16, '19):

The sequence $\frac{1}{n} K(L_n)$ satisfy a LDP with a convex rate function I.

(Le Page $\overset{82}{\Rightarrow}$ $\vec{\lambda}_\mu$ is the unique zero of I)

$\to \{\alpha \mid I(\alpha) < \infty\} \underset{\text{up to bounding points.}}{=} J(S)$



<u>Conjecture</u>: $\frac{1}{n} \lambda(L_n)$ satisfy the same LDP.