Conjugacy growth in groups, geometry and combinatorics

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Séminaire de Groupes et Géométrie, Geneva, May 19, 2020

EPSRC
Engineering and Physical Sciences
Research Council

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Counting elements and conjugacy classes in groups

Let G be a group with finite generating set X.

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- Standard growth of G: number of elements of length n in G, for all $n \geq 0$.
- Conjugacy growth of G: number of conjugacy classes containing an element of length n in G, for all $n \geq 0$.

Counting in groups

1. Standard growth functions:

$$\texttt{sphere} \longrightarrow a(n) = a_{G,X}(n) := \sharp\{g \in G \mid |g|_X = n\}$$

$$\texttt{ball} \longrightarrow A(n) = A_{G,X}(n) := \sharp\{g \in G \mid |g|_X \leq n\}.$$

2. Conjugacy growth functions:

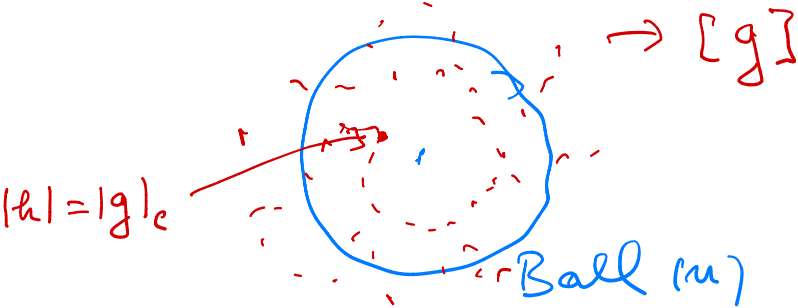
$$c(n) = c_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c = n\}$$

$$C(n) = C_{G,X}(n) := \sharp\{[g] \in G \mid |g|_c \leq n\},$$

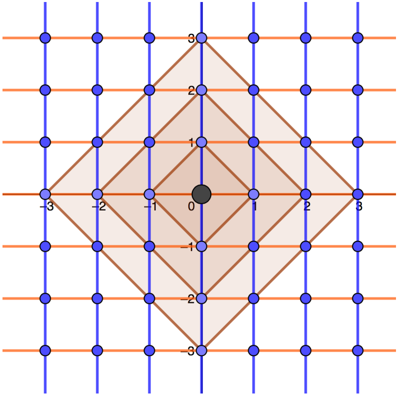
where $|g|_c$ is the length of a shortest element in the conjugacy class $[g]$, with respect to X.

Counting conjugacy classes in groups

The conjugacy length $|g|_c$ of $[g]$ is the length of a shortest element in the conjugacy class $[g]$, with respect to X.

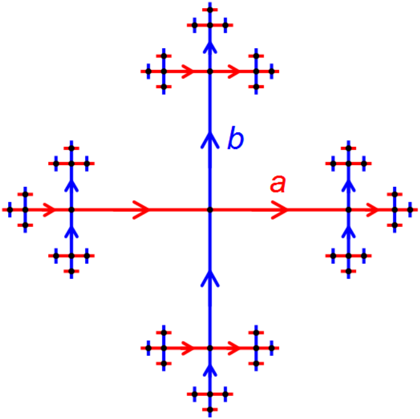


$\mathbb{Z}^2$ with standard generators a and b



$$a(k) = c(k) = 4k, \quad A(n) = C(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

Examples: F_2 with free generators a and b



$$a(n) = 4 \cdot 3^{n-1}$$

Asymptotics of conjugacy growth in the free group F_r

Idea: take all cyclically reduced words of length n, and divide by n.

$$\underline{a^2b} \sim aba \sim ba^2$$

$\uparrow$

Asymptotics of conjugacy growth in the free group F_r

Essentially, take all cyclically reduced words of length n, whose number is $\sim (2r-1)^n$, and divide by n.

abab → baba

$x_1 x_2 \ldots x_n \sim x_2 \ldots x_n x_1 \sim$

Not entirely correct: when powers are included, one shouldn't divide by n.

Asymptotics of conjugacy growth in the free group F_r

Coornaert (2005): For the free group F_r, the primitive conjugacy growth function (of non-power elements) is given by

$$c_p(n) \sim \frac{(2r-1)^{n+1}}{2(r-1)n} = \kappa_1 \frac{(2r-1)^n}{n}.$$

Since the density of powers in any free group is 0, we get that

$$c(n) \sim \kappa_2 \frac{(2r-1)^n}{n} \sim \kappa_3 \frac{a(n)}{n}.$$

Conjugacy growth: history and motivation

Conjugacy growth in geometry

Counting the primitive closed geodesics of bounded length on a compact manifold M of negative curvature and exponential volume growth gives

via quasi-isometries

good exponential asymptotics for the primitive conjugacy growth of the fundamental group of M.

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- 1960s (Sinai, Margulis): $M=$ complete Riemannian manifolds or compact manifolds of pinched negative curvature;
- 1990s - 2000s (Knieper, Coornaert, Link): some classes of (rel) hyperbolic or CAT(0) groups.

Conjugacy growth asymptotics

- Babenko (1989): asymptotics for virtually abelian and the discrete Heisenberg groups.
- Rivin (2000), Coornaert (2005): asymptotics for the free groups.

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- Rivin (2000), Coornaert (2005): asymptotics for the free groups.
- Guba-Sapir (2010): asymptotics for some Baumslag-Solitar groups and other HNN-extensions, diagram groups etc.
- **Conjecture (Guba-Sapir)**: groups* of standard exponential growth have exponential conjugacy growth.

Conjugacy growth asymptotics

- Breuillard-Cornulier (2010): uniform exponential conjugacy growth for f.g. solvable (non virt. nilpotent) groups.
- Breuillard-Cornulier-Lubotzky-Meiri (2011): uniform exponential conjugacy growth for f.g. linear (non virt. nilpotent) groups.
- Hull-Osin (2014): all acylindrically hyperbolic groups have exponential conjugacy growth.

What is the relation between $A(n)$ and $C(n)$?

C(n) vs A(n)??

- Easy (no partial credit): $C(n) \leq A(n)$ and $C(n) = A(n)$ for abelian groups.
- Medium:

$$\limsup_{n\to\infty} \frac{C(n)}{A(n)} = ?$$

→ > 0 virt. ab.

→ = 0 free gps.

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- Hard: f.p. / amenable

  Conjecture (Guba-Sapir): groups* of standard exponential growth have exponential conjugacy growth.

  ⋆ Exclude the Osin or Ivanov type 'monsters'!

- Easy/Hard: Compare standard and conjugacy growth rates.

Growth rates

The standard growth rate of G wrt X, defined as $\alpha_{G,X} = \limsup_{n\to\infty} \sqrt[n]{a(n)}$, is in fact a limit:

$$\limsup_{n\to\infty} \sqrt[n]{a(n)} = \lim_{n\to\infty} \sqrt[n]{a(n)}.$$

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The conjugacy growth rate of G wrt X can be defined as

$$\gamma_{G,X} = \limsup_{n\to\infty} \sqrt[n]{c(n)}.$$

Hull: There are groups for which

$$\liminf_{n\to\infty} \sqrt[n]{c(n)} < \limsup_{n\to\infty} \sqrt[n]{c(n)},$$

that is, the limit does not exist.

Conjugacy vs. standard growth

	Standard growth	Conjugacy growth
Type	pol., int., exp.	pol., int.*, exp.
Quasi-isometry invariant	yes	no**, but group invariant
Rate of growth	exists	exists (not always)

* Bartholdi, Bondarenko, Fink.

** Hull-Osin (2013): conjugacy growth not quasi-isometry invariant.

The conjugacy growth series

The conjugacy growth series

Let G be a group with finite generating set X.

- The conjugacy growth series of G with respect to X records the number of conjugacy classes of every length. It is

$$\sigma_{(G,X)}(z) := \sum_{n=0}^{\infty} c(n) z^n,$$

where $c(n)$ is the number of conjugacy classes of length n.

Growth rates from power series

- For a complex series $f(z) = \sum_{i=0}^{\infty} a_i z^i$ the radius of convergence satisfies:

$$RC(f) = \frac{1}{\limsup_{i\to\infty} \sqrt[i]{a_i}} = \frac{1}{\alpha}.$$

- For any rational function $f(z) = \frac{P(z)}{Q(z)}$ the radius of convergence $RC(f)$ of f is the smallest absolute value of a zero of $Q(z)$.

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Rational conjugacy growth series give conjugacy asymptotics.

Question: For which groups are conjugacy growth series rational?

Rational, algebraic, transcendental

A generating function $f(z)$ is

- rational if there exist polynomials $P(z)$, $Q(z)$ with integer coefficients such that $f(z) = \frac{P(z)}{Q(z)}$;
- algebraic if there exists a polynomial $P(x, y)$ with integer coefficients such that $P(z, f(z)) = 0$;
- transcendental otherwise.

Rationality

Having a rational standard growth series is not a group invariant!

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Theorem [Stoll, 1996]

The higher Heisenberg groups H_r have rational growth with respect to one choice of generating set and transcendental with respect to another.

$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c, d, e \in \mathbb{Z} \right\}$$

Conjugacy growth series: hyperbolic groups

Virtually cyclic groups: $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is the same as the standard one:

$$\sigma_{(\mathbb{Z},\{1,-1\})}(z) = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\sigma_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1+2z-2z^3}{1-z^2}.$$

The conjugacy growth series in free groups

- Rivin (2000, 2010): the conjugacy growth series of F_k is not rational:

$$\sigma(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

Conjecture (Rivin, 2000)

If G hyperbolic, then the conjugacy growth series of G is rational if and only if G is virtually cyclic.

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$\Rightarrow$

Theorem (Antolín - C., 2017)

If G is non-elementary hyperbolic, then the conjugacy growth series is transcendental.

$\Leftarrow$

Theorem (C. - Hermiller - Holt - Rees, 2016)

Let G be a virtually cyclic group. Then the conjugacy growth series of G is rational.

NB: Both results hold for all symmetric generating sets of G.

Idea of proof: Asymptotics of conjugacy growth in hyperbolic groups

Theorem. (Coornaert - Knieper 2007, Antolín - C. 2017)

Let G be a non-elementary word hyperbolic group. Then there are positive constants A, B and n_0 such that

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

for all $n \geq n_0$, where α is the growth rate of G.

Idea of proof: Asymptotics of conjugacy growth in hyperbolic groups

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MESSAGE:

The number of conjugacy classes in the ball of radius n is asymptotically the number of elements in the ball of radius n divided by n.

End of the proof: Analytic combinatorics at work

The transcendence of the conjugacy growth series for non-elementary hyperbolic groups follows from the bounds

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

End of the proof: Analytic combinatorics at work

The transcendence of the conjugacy growth series for non-elementary hyperbolic groups follows from the bounds

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

together with

Lemma (Flajolet: Trancendence of series based on bounds).

Suppose there are positive constants $A, B, \mathbf{h}$ and an integer $n_0 \geq 0$ s.t.

$$A\frac{e^{\mathbf{h}n}}{n} \leq a_n \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$. Then the power series $\sum_{i=0}^{\infty} a_n z^n$ is not algebraic.

Consequence of Rivin's Conjecture

Corollary (Antolín - C.)

For any hyperbolic group G with generating set X the standard and conjugacy growth rates are the same:

$$\lim_{n\to\infty} \sqrt[n]{c(n)} = \gamma_{G,X} = \alpha_{G,X}.$$

$$\sqrt[n]{c(n)} \sim \sqrt[n]{\frac{a(n)}{n}}$$

$$\gamma = \alpha$$

Acylindrically hyperbolic groups

Main Theorem (Antolín - C., 2017)

Let G be an acylindrically hyperbolic group, X any finite generating set, and $\mathcal{L}_c$ be a set containing one minimal length representative of each conjugacy class.

Then $\mathcal{L}_c$ is not unambiguous context-free, so not regular.

Rivin's conjecture for other groups

Theorem (Gekhtman and Yang, 2019)

Let G be a non-elementary group with a finite generating set S. If G has a contracting element with respect to the action on the corresponding Cayley graph, then the conjugacy growth series is transcendental.

Examples: relatively hyperbolic groups, (non-abelian) RAAGs, RAACs, graph products of finite groups, graphical small cancellation groups.

Contracting elements

$G \curvearrowright X$ (= Cay(G)?), $\sigma \in X$

$g \in G$ is a c.e. if ① $\langle g \rangle \cdot \sigma \xrightarrow{q.e.} X$

② $\langle g \rangle \cdot \sigma$ is a contracting set

$G \curvearrowright$ Cay(G).

G hyp. : any infinite order element is contracting.

$\Rightarrow$ conj. growth series is transc

Rationality of standard and conjugacy growth series

	Standard Growth Series	Conjugacy Growth Series
Hyperbolic	Rational (Cannon, Gromov, Thurston)	Transcendental (Antolín - C. '17)
Virtually abelian	Rational (Benson '83)	Rational (Evetts '19)
Heisenberg H_1	Rational (Duchin-Shapiro '19)	Transcendental

FOR ALL GENERATING SETS!

→ $n^2 \log n$

Standard generating set ... for now

	Conjugacy Growth Series	Formula
Wreath products	Transcendental (Mercier '17)	✓
Permutational wreath products	Transcendental (Bacher, de la Harpe '17)	✓
Graph products	Transcendental[1] (C.- Hermiler - Mercier '19)	✓
BS(1,m)	Transcendental (C.- Evetts - Ho, '20)	✓

[1] depends on vertex groups

The combinatorics at the core of conjugacy growth

- Let L be a set of words, $a(k)$ the number of words of length k in L, and $f_L(t) = \sum_{k\geq 1} a(k)t^k$ the generating function of L.

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  Assume L is closed under taking powers and cyclic permutations of words.

- The generating function for the language $L_\sim$ of cyclic representatives is

$$\int_0^z \frac{\sum_{k \geq 1} \phi(k) f_L(t^k)}{t} \, dt,$$

[$a^2b \sim ba^2 \sim aba$]

and the growth rates of L and $L_\sim$ are the same.

Rivin's formula for free groups

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After some computation the general formula for languages

$$\int_0^z \frac{\sum_{d\geq 1} \phi(d) f_L(t^d)}{t}\, dt$$

agrees with Rivin's formula

$$\sigma(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

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Questions

$$c(n) \sim \frac{a(n)}{n^2}$$

- Are the standard and conjugacy growth rates equal for all ‘reasonable’ groups?

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  Stoll: The rationality of the standard growth series depends on generators.

Merci!