CO-WORD PROBLEMS AND GEODESIC GROWTH IN FINITELY GENERATED GROUPS

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PLAN

Part 1: formal languages

Part 2: co-word problem of bounded automata groups

Part 3: geodesic growth

Recall: a context-free grammar is a tuple (N, Σ, R, S) where

- N, Σ are finite alphabets
- R is a finite set of productions $A \to \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$
- $S \in N$ is the *start symbol*

Eg: $S \to SS, \;\; S \to aSb, \;\; S \to \epsilon$ produces the language:

A pushdown automaton (PDA) is a finite state automaton plus a FILO stack:

Eg: $L = \{a^n b^n\}$
Fact: $L \subseteq \Sigma^*$ is the language of a context-free grammar iff L is the language of strings accepted by some PDA.

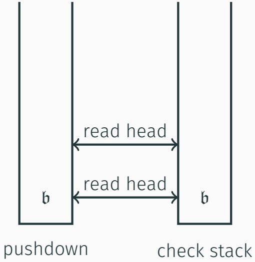
ET0L, CSPD AUTOMATON

An ET0L grammar is similar to a context-free one, except

- rules are grouped into subsets called *tables*
- when you apply a table, *every* symbol from N must be changed in parallel.

Eg: $f = \{S \to ABA, A \to aA, B \to bB\}, \;\; g = \{S \to S, A \to \epsilon, B \to \epsilon\}$
produces the language:

A check stack pushdown automaton (CSPD) is a finite state automaton plus two stacks:



Phase 1: load up the check stack.

Phase 2: check stack is read-only. Eg: $a^n b^n$

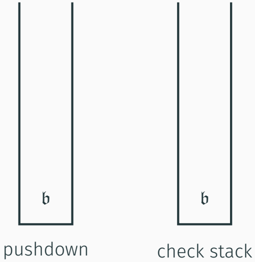
Fact[1]: $L \subseteq \Sigma^*$ is the language of an ET0L grammar iff L is the ~~language of strings accepted by~~ some CSPD.

[1] Leeuwen, “Variations of a new machine model”, 1976.

ET0L, CSPD AUTOMATON

Eg: $L = \{ab^{i_1}ab^{i_2}\dots ab^{i_n} \mid i_1 \leq i_2 \leq \dots \leq i_n\}$ has intermediate growth (as a set of words), and is indexed[2].

But actually its ET0L:



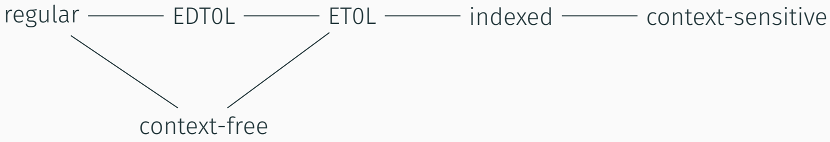
(In fact it is EDT0L, grammar in[3].)

[2] Grigorchuk and Machì, “An example of an indexed language of intermediate growth”, 1999.

[3] Ciobanu, Elder, and Ferov, “Applications of L systems to group theory”, 2018.

context-free $\subset$ ET0L:

Make every table include $A \to A$ for every $A \in N$.



Ciobanu, Diekert, E[4,5,6]: solution sets as tuples of reduced/geodesic/shortlex words to equations in

- free groups
- virtually free groups
- hyperbolic groups
- ...

are EDT0L.

[4] Ciobanu, Diekert, and Elder, "Solution sets for equations over free groups are EDT0L languages", 2016.
[5] Diekert and Elder, "Solutions to twisted word equations and equations in virtually free groups", 2020.
[6] Ciobanu and Elder, "Solutions sets to systems of equations in hyperbolic groups are EDT0L in PSPACE", 2019.

Definition: Co-Word Problem

Given a group G with finite symmetric generating set X, the co-word problem

$$coW(G, X) = \{w \in X^* \ : \ \overline{w} \neq 1_G\}$$

is the set of all words that don't represent the identity $1_G \in G$.

Eg: $\mathbb{Z} = \langle a \rangle$ and $X = \{a, a^{-1}\}$

A group has a

- regular co-word problem iff it's finite[7]
- deterministic context-free co-word problem iff it's virt. free[8]
- deterministic n-counter co-word problem iff it's virt. $\mathbb{Z}^n$ [9]

Also,

- Thompson's group V has a context-free co-word problem[10]
- Bounded automata groups have indexed co-word problem[11]

[7] Anisimov, "The group languages", 1971.

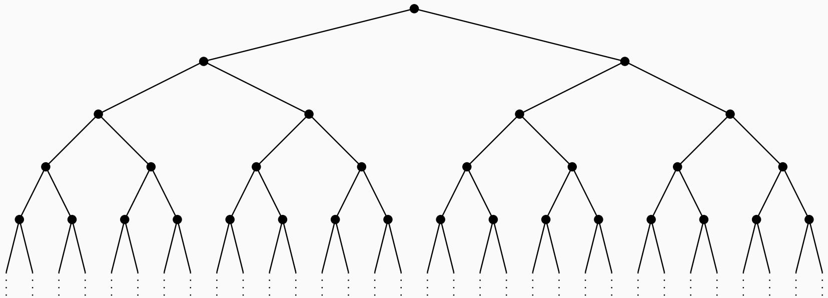
[8] Muller and Schupp, "Groups, the theory of ends, and context-free languages", 1983.

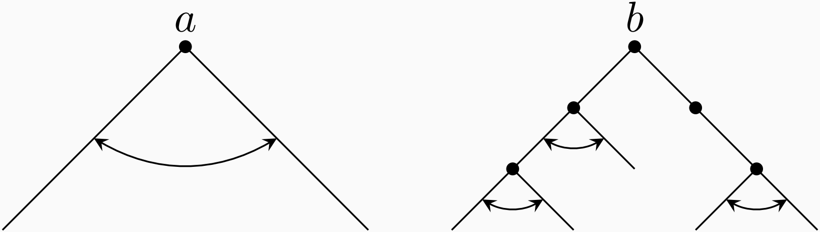
[9] Elder, Kambites, and Ostheimer, "On groups and counter automata", 2008.

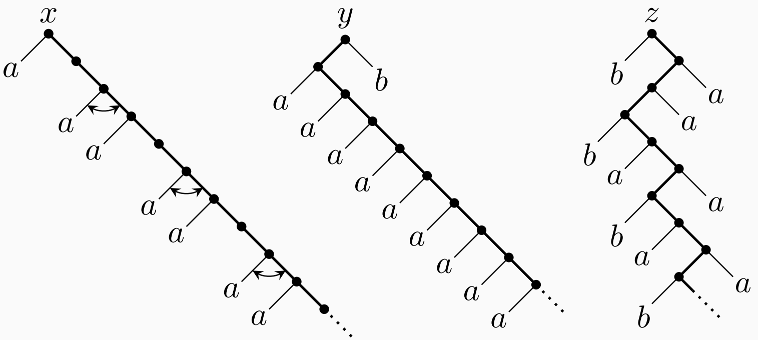
[10] Lehnert and Schweitzer, "The co-word problem for the Higman-Thompson group is context-free", 2007.

[11] Holt and Röver, "Groups with indexed co-word problem", 2006.

BOUNDED AUTOMATA GROUPS




a
b



We demand that this *spine* is eventually periodic

Definition (Bounded Automaton Automorphism)

An automorphism is *bounded automaton* if it can be expressed by composing (finitely many) finitary and directed automorphisms.

Definition (Bounded Automata Group)

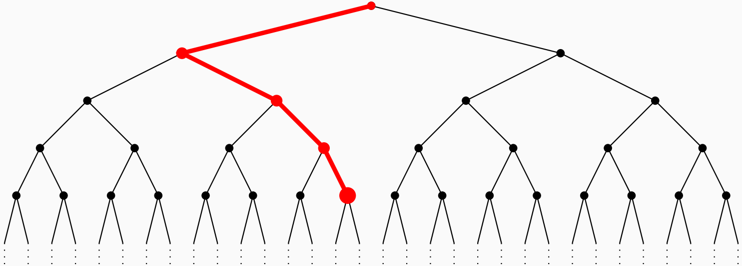
A group is *bounded automata* if it has a generating set of bounded automaton automorphisms.

RESULT

Theorem (Bishop-E; 2019[12])

Every finitely generated bounded automata group has an ET0L co-word problem

Proof idea: A word $w = w_1 w_2 \cdots w_k$ is in the co-word problem iff its action moves some vertex, e.g.,



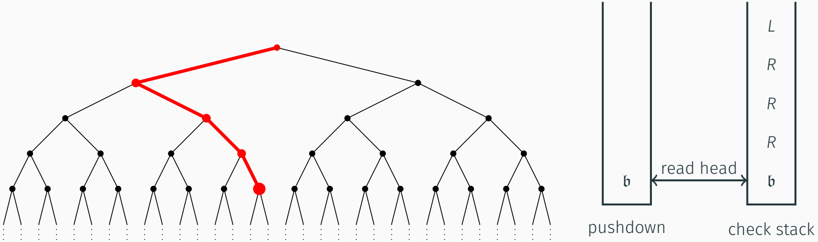
[12] Bishop and Elder, "Bounded automata groups are co-ET0L", 2019.

Step 1: load up the check stack with the address of the vertex

Step 2: copy the address over to the pushdown stack.

Step 3: each input letter (automoprhism) moves the vertex on the pushdown – rewrite pushdown

End: check the vertex in the pushdown is not the same as the check stack vertex



so..

This improves on Holt and Röver who showed indexed

Note:[13] gave an ET0L grammar for the first Grigrochuk group, but complicated and hard to generalise to arbitrary bounded automata groups

[13] Ciobanu, Elder, and Ferov, “Applications of L systems to group theory”, 2018.

Definition: Geodesic growth

Given a group G with finite symmetric generating set X, the function $\gamma_{G,X} : \mathbb{N} \to \mathbb{N}$ defined by

$$\gamma_{G,X}(n) = \#\left\{\sigma \in X^* \mid |\sigma|_X \leq n \text{ and } \sigma \text{ is geodesic}\right\}.$$

is called the geodesic growth function for G with respect to X.

Eg: $\mathbb{Z} = \langle a \mid - \rangle$ and $X = \{a, a^{-1}\}$

Eg: $\mathbb{Z} = \langle a, b \mid a = b \rangle$ and $X = \{a, b, a^{-1}, b^{-1}\}$

We say that, with respect to the generating set X, the group G has

- *exponential geodesic growth* if there exists $\alpha > 1$ such that $\gamma_{G,X}(n) \geq \alpha^n$;
- *polynomial geodesic growth* if there exists $\beta, d \in \mathbb{R}$ such that $\gamma_{G,X}(n) \leq \beta n^d$;and
- *intermediate geodesic growth* otherwise.

Note

$$\text{usual growth} \leq \gamma_{G,X} \leq |X|^n$$

1. Does there exist a group G with finite generating set X such that $\gamma_{G,X}$ is intermediate?

2. Characterise those groups G which have polynomial geodesic growth with respect to some finite symmetric generating set

$$\text{usual growth} \leq \gamma_{G,X} \leq |X|^n$$

Theorem (Shapiro; 1997[14])

$\mathbb{Z}^2$ has exponential geodesic growth with respect to every finite generating set.

Theorem (Bridson-Burillo-E-Šunić; 2012)

Let G be a finitely generated nilpotent group. If G is not virtually cyclic, then G has exponential geodesic growth with respect to every finite generating set.

Note: If G is virtually cyclic, it is hyperbolic, and the full set of geodesics is a regular language[15]. Thus G can have either polynomial or exponential geodesic growth only.

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[14] Shapiro, "Pascal's triangles in abelian and hyperbolic groups", 1997.

[15] Cannon, "The combinatorial structure of cocompact discrete hyperbolic groups", 1984.

VIRTUALLY ABELIAN WITH POLYNOMIAL GEODESIC GROWTH

Let $G = \langle a, b, t \mid ab = ba, t^2 = 1, tat = b \rangle$ (example of Cannon).

Clearly virtually $\mathbb{Z}^2$. Delete the b with a Tietze transformation:

$\langle a, t \mid t^2 = 1, atat = tata \rangle$.

(picture)

A word which jumps between copies more than twice, (*i.e.* has 3 or more ts), is not geodesic.

So G is virtually $\mathbb{Z}^2$ and has polynomial geodesic growth with respect to $X = \{a, a^{-1}, t\}$.

Theorem (Bishop; 2020[16])

If G is virtually abelian, then G can have either polynomial or exponential geodesic growth only (any generating set)

The proof is quite involved: bijection between geodesics and a certain kind of formal language called linearly constrained[17]

Massazza[18]: generating function for a linearly constrained language is D-finite (holonomic), which implies that no such language has intermediate growth.

[16] Bishop, "Geodesic growth in virtually abelian groups", 2020.

[17] intersection of an unambiguous context-free language with a language of words satisfying a finite set of linear constraints on the number of occurrences of symbols

[18] Massazza, "Holonomic functions and their relation to linearly constrained languages", 1993.

Theorem (Bridson-Burillo-E-Šunić; 2012)

Let G be a finitely generated group. If there exists $x \in G$ whose normal closure is abelian and of finite index, then there exists a finite generating set wrt which G has polynomial geodesic growth.

So we know virtually abelian groups can have polynomial geodesic growth

but is that all there is?

Let *vH* be two copies of the discrete Heisenberg group

$$\langle a, b \mid [[a, b], a] = 1, [[a, b], b] = 1\rangle$$

glued together using the same trick as above:

$$vH = \left\langle a, b, t \mid [a, [a, b]] = [b, [a, b]] = t^2 = 1,\ a^t = b \right\rangle$$

$$\left\langle a, t \mid [a, [a, a^t]] = [a^t, [a, a^t]] = t^2 = 1 \right\rangle$$

Theorem (Bishop-E; 2020[19])

The group vH is virtually nilpotent of step 2, and has polynomial geodesic growth with respect to the generating set $\{a, a^{-1}, t\}$.

[19] Bishop and Elder, "A virtually Heisenberg group with polynomial geodesic growth", 2020.

Key to the proof:

Blachère[20] calculated an explicit length formula for H with respect to the generating set $\{a, a^{-1}, b, b^{-1}\}$.

Fact arising from this: every element of H has at least one geodesic representative of the form

$$a^{i_1} b^{i_2} a^{i_3} b^{i_4} a^{i_5} b^{i_6} \quad \text{or} \quad b^{i_1} a^{i_2} b^{i_3} a^{i_4} b^{i_5} a^{i_6}$$

In vH, with generating set $\{a, a^{-1}, t\}$, this means you can express any word by swapping between copies of H at most 7 times.

[20] Blachère, "Word distance on the discrete Heisenberg group", 2003.

Questions:

- Maybe only step 2?
- Another trick to get intermediate?

Thanks!

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