Rigid measures on G-spaces

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Rigid measures: definition

Definition

Let G be a locally compact second countable (lcsc) group, and let X be a compact G-space. A non-singular probability measure $\nu \in \mathrm{Prob}(X)$ is said to be *G-rigid* if it has full support and the canonical embedding is the unique G-equivariant unital positive map from $C(X)$ into $L^\infty(X, \nu)$.

Remark

1. It follows from results of Ozawa (2007) and Bassi–Radulescu (2020) that if $\nu \in \mathrm{Prob}(X)$ is G-rigid then $\mathrm{id} : X \to X$ has the alignment property in the sense of Furman (JAMS 2008):
an equivariant measurable map $\pi : X \to B$, where (X, ν) is a non-singular probability G-space, B is a topological space, is said to have the alignment property if the only equivariant map from $X \to \mathrm{Prob}(B)$ is the one given by $x \mapsto \delta_{\pi(x)}$.

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Remark

2. The requirement of ν having full support is of course not necessary in order to make sense of the definition. But it suffices for the purpose of this talk and simplifies some statements.

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Remark

3. Observe from the definition that rigidity is in fact a property of the measure-class rather than a single measure. Furthermore, it can be considered as a property of the algebra $L^\infty(X, \nu)$ - the existence of an L^1-dense (L^∞-closed) algebra with unique equivariant u.p. map.

Rigid measures vs. boundaries

G-spaces supporting rigid measures share many properties with G-boundaries.

Let $G \curvearrowright X$, and $\mu \in \mathrm{Prob}(G)$ be absolutely continuous,

1. **Topological boundary actions:** we say X is a G-boundary if for every $\nu \in \mathrm{Prob}(X)$ and $x \in X$ there are $(g_i) \subset G$ such that $g_i\nu \to \delta_x$.

2. **Measure-theoretical boundary actions:** a probability $\nu \in \mathrm{Prob}(X)$ is said to be μ-stationary if $\mu * \nu = \nu$. The μ-stationary measure ν is said to be a μ-boundary if for almost every trajectory $\omega = (g_k) \in \Omega$ of the μ-random walk on G, the sequence $(g_k\nu)$ converges to a Dirac measure δ_{x_ω}.

Some elementary facts

Proposition

If a compact G-space X admits both a rigid probability measure and an invariant probability measure then X is a singleton.

Thus, if G admits a non-trivial action with a rigid measure, then G is non-amenable;

and, the only finite G-space that admits a rigid measure is the trivial one.

Proposition

Let $G \curvearrowright X$, and $\nu \in \mathrm{Prob}(X)$ be G-rigid. Then the only bounded linear G-equivariant map on $L^1(X, \nu)$, sending probs to probs, is the identity map.

General properties

Proposition

Let $G \curvearrowright X$ and let $\nu \in \mathrm{Prob}(X)$ be G-rigid. Then every element $g \in G$ whose centralizer $C_G(g)$ has finite co-volume in G acts trivially on X.

Corollary

If a discrete group Γ admits a faithful action on a compact space X supporting a rigid measure, then Γ is an ICC group.
(The converse is not true, as there are amenable ICC groups.)

Restriction to lattices

Theorem

Suppose $H \leq G$ is a closed subgroup with finite co-volume (e.g. a lattice). Then any G-rigid measure ν on any compact G-space X is H-rigid.

Rigid measures: Examples

Theorem

Let $\mu \in \mathrm{Prob}(G)$ be an absolutely continuous probability on G. Let X be a compact G-space supporting a unique μ-stationary probability ν such that ν is a μ-boundary. Then ν is G-rigid.

Rigid measures: Examples

Example

For the following actions $\Gamma \curvearrowright X$, any generating measure on G satisfies the above:

- linear groups acting on flag varieties (Ledrappier; Kaimanovich; Brofferio-Schapira);
- hyperbolic groups acting on the Gromov boundary (Kaimanovich);
- non-elementary subgroups of mapping class groups acting on the Thurston boundary (Kaimanovich-Masur);
- non-elementary subgroups of $\mathrm{Out}(\mathbb{F}_n)$ acting on the boundary of the outer space (Horbez).
- ...

Rigid measures vs. topological boundaries

Proposition (Ozawa; Bassi–Radulescu)

Let Γ be a discrete group, $\Gamma \curvearrowright X$, and $\nu \in \mathrm{Prob}(X)$ be Γ-rigid. If the action $\Gamma \curvearrowright (X, \nu)$ is Zimmer-amenable then X is a Γ-boundary.

Recall: $G \curvearrowright (Y, \eta)$ is Zimmer amenable if there is an equivariant positive linear projection $L^\infty(G \times Y) \to L^\infty(Y)$.

Rigid measures vs. topological boundaries

(The argument given in proof of the above result does not directly generalize to non-discrete group case, but) we have:

Theorem

Let G be a l.c.s.c. group which contains a lattice. Let $G \curvearrowright X$, and $\nu \in \mathrm{Prob}(X)$ be G-rigid. If $G \curvearrowright (X, \nu)$ is Zimmer amenable then X is a G-boundary.

Rigid measures: examples

Suppose $\Gamma \curvearrowright X$, and $\Lambda \leq \Gamma$ such that

- there is a unique Λ-invariant probability on X of the form δ_{x_0} for a Λ-fixed point $x_0 \in X$,
- the Γ-orbit of x_0 is dense in X,

then for any fully supported $\mu \in \mathrm{Prob}(\Gamma)$, the measure $\nu = \sum_{g\in\Gamma} \mu(g)\delta_{gx_0}$ is rigid.

e.g. $\Gamma \curvearrowright X$ convergence action, $g \in \Gamma$ a parabolic element.

$\Gamma \curvearrowright X$ minimal, X is metrizable, $g \in G$ Lipschitz, with constant < 1.

Rigid measures: examples

Say the subgroup $\Lambda \leq \Gamma$ has *the spectral gap property* if $\Lambda \curvearrowright (\Gamma/\Lambda) \setminus \{\Lambda\}$ has spectral gap.
This is equivalent to that $\delta_{\{\Lambda\}}$ being the unique Λ-invariant mean on $\ell^\infty(\Gamma/\Lambda)$.

If $\Lambda \leq \Gamma$ has the spectral gap property, then $\Gamma \curvearrowright \beta(\Gamma/\Lambda)$ satisfies the above, hence any $\nu \in \mathrm{Prob}(\Gamma/\Lambda)$ with full support is rigid as a probability on $\beta(\Gamma/\Lambda)$.

e.g. $\Gamma = SL_{n+1}(\mathbb{Z})$, $\Lambda = SL_n(\mathbb{Z})$ for $n \geq 2$
$\Gamma = \Lambda * \Upsilon$, Λ is non-amenable, $|\Upsilon| \geq 3$.

Rigid measures: applications

Representation rigidity of subgroups

For $\Lambda \leq \Gamma$, denote by $\lambda_{\Gamma/\Lambda}$ the quasi-regular representation of Γ on $\ell^2(\Gamma/\Lambda)$.

By Mackey, $\lambda_{\Gamma/\Lambda}$ is irreducible iff Λ is self-commensurated in Γ, that is

$$\{g \in \Gamma \mid [\Lambda : \Lambda \cap g\Lambda g^{-1}] < \infty \text{ and } [g\Lambda g^{-1} : \Lambda \cap g\Lambda g^{-1}] < \infty\} = \Lambda.$$

Theorem (Mackey 1951)

Let Λ and Υ be self-commensurated subgroups of Γ. If $\lambda_{\Gamma/\Lambda} \approx_u \lambda_{\Gamma/\Upsilon}$, then Λ is conjugate to Υ.

Rigid measures: applications

But the more appropriate notion of equivalence in discrete case is *weak equivalence*.

Representation rigidity of subgroups

Let $\pi, \sigma \in \mathrm{Rep}(\Gamma)$. Recall that π is weakly contained in σ, written $\pi \prec \sigma$, if

$$\| \sum_{i=1}^{n} c_i \pi(g_i) \|_{B(H_\pi)} \leq \| \sum_{i=1}^{n} c_i \sigma(g_i) \|_{B(H_\sigma)}$$

for any $c_1, \ldots, c_n \in \mathbb{C}$, $g_1, \ldots, g_n \in \Gamma$, $n \in \mathbb{N}$.

We say π is weakly equivalent to σ, written $\pi \approx \sigma$, if $\pi \prec \sigma$ and $\sigma \prec \pi$.

Theorem (Bekka-K.)

Let $\Lambda \leq \Gamma$ have the spectral gap property, and let $\Upsilon \leq \Gamma$ be self-commensurated. If $\lambda_{\Gamma/\Lambda} \approx \lambda_{\Gamma/\Upsilon}$, then Λ is conjugate to Υ.

Rigid measures: applications

Representation rigidity of subgroups

$\pi \prec \sigma$ is equivalent to say that the map $\sigma(g) \mapsto \pi(g)$ extends to a continuous map φ from the norm-closure $C^*_\sigma(\Gamma)$ of the $\mathrm{span}\{\sigma(g) : g \in \Gamma\}$ to the norm-closure $C^*_\pi(\Gamma)$ of the $\mathrm{span}\{\pi(g) : g \in \Gamma\}$.

In this case, by Arveson's extension theorem, the map φ extends to a completely positive map $B(H_\sigma) \to B(H_\pi)$, which would automatically be Γ-equivariant for the action of Γ by inner automorphisms on both sides.

Rigid measures: applications

Representation rigidity of subgroups

But existence a Γ-equivariant unital completely positive map $B(H_\sigma) \to B(H_\pi)$ is a much weaker condition than $\pi \prec \sigma$.
Let us write $\pi \prec\prec \sigma$ for this property. Then $1_\Gamma \prec\prec \sigma$ for any σ, and $\sigma \prec\prec 1_\Gamma$ iff σ is amenable in the sense of Bekka.
Thus, $\prec\prec$ gives rise to an equivalence relation in which all amenable representations are equivalent.

Theorem

Let Λ and Υ be subgroups of Γ with the spectral gap property. If $\lambda_{\Gamma/\Lambda} \prec\prec \lambda_{\Gamma/\Upsilon}$ and $\lambda_{\Gamma/\Upsilon} \prec\prec \lambda_{\Gamma/\Lambda}$, then Λ is conjugate to Υ.

Rigid measures: applications

Theorem (Nevo–Sageev 2013)

Let Γ *be a countable group and* $\mu \in \mathrm{Prob}(\Gamma)$ *be generating such that the Poisson boundary of* (Γ, μ) *has a uniquely stationary compact model* (B, ν)*. Then any Zimmer-amenable* μ*-stationary action of* Γ *is a measurable extension of* (B, ν)*.*

Rigid measures: applications

Theorem (Nevo–Sageev 2013)

Let Γ be a countable group and $\mu \in \mathrm{Prob}(\Gamma)$ be generating such that the Poisson boundary of (Γ, μ) has a uniquely stationary compact model (B, ν). Then (B, ν) has no proper Zimmer-amenable Γ-factor.

Rigid measures: applications

Generalizations (commutative case):

Theorem

Let $\Gamma \curvearrowright X$, $\nu \in \mathrm{Prob}(X)$ *be* Γ*-rigid, and* $\Gamma \curvearrowright (Z, m)$ *be a p.m.p. action. Then there is no proper Zimmer-amenable intermediate factor* $(X \times Z, \nu \times m) \to (Y, \eta) \to (Z, m)$.

Rigid measures: applications

von Neumann algebras

Recall that a von Neumann algebra is unital self-adjoint subalgebra $M \subseteq B(H)$ which is closed in weak operator topology.

e.g. the weak closure $L\Gamma$ of $\mathrm{span}\{\lambda_g : g \in \Gamma\} \subset B(\ell^2(\Gamma))$

$\Gamma \curvearrowright (Y, \eta)$ non-singular $\rightsquigarrow$ *the crossed product von Neumann algebra* $\Gamma \ltimes L^\infty(Y, \eta)$ is the weak closure of

$$\mathrm{span}\{\tilde{\lambda}_g \pi(f) : g \in \Gamma, f \in L^\infty(Y, \eta)\} \subset B(L^2(\Gamma \times X)),$$

where $\tilde{\lambda}_g(\xi)(h, x) = \xi(g^{-1}h, x)$ and $[\pi(f)(\xi)](h, x) = f(hx)\xi(h, x)$

(main point: $\Gamma \ltimes L^\infty(Y, \eta)$ contains a copy of $L\Gamma$ and a copy of $L^\infty(Y, \eta)$, and $\pi(f_g) = \tilde{\lambda}_g \pi(f) \tilde{\lambda}_{g^{-1}}$

Rigid measures: applications

Amenable von Neumann algebras

There is a natural notion of amenability for von Neumann algebras:

we have $L\Gamma$ is amenable iff Γ is amenable, and

$\Gamma \ltimes L^\infty(Y, \eta)$ is amenable iff $\Gamma \curvearrowright (Y, \eta)$ is Zimmer amenable.

Theorem

Let $\Gamma \curvearrowright X$, $\nu \in \mathrm{Prob}(X)$ *be* Γ*-rigid, and let* $\Gamma \curvearrowright (Z, m)$ *be a p.m.p. action. Suppose*

$$\Gamma \ltimes L^\infty(Z, m) \subseteq M \subseteq \Gamma \ltimes L^\infty(X \times Z, \nu \times m)$$

is an inclusion of von Neumann algebras. If M *is amenable, then* $M = \Gamma \ltimes L^\infty(X \times Z, \nu \times m)$.

Rigid measures: applications

Maximal amenable von Neumann subalgebras

An important problem in von Neumann algebras: Given a vN algebra M, describe its maximal amenable vN subalgebras;
(e.g., is $L\Lambda$ a maximal amenable subalgebra of $L\Gamma$ if $\Lambda \leq \Gamma$ is maximal amenable subgroup?)

A general fact for vN algebras $M \subseteq B(H)$:
M is amenable iff $M' := \{T \in B(H) : ST = TS \text{ for all } S \in M\}$ is amenable.

Rigid measures: applications

Maximal amenable von Neumann subalgebras

An important problem in von Neumann algebras: Given a vN algebra M, describe its maximal amenable vN subalgebras.

Theorem

Let $\Gamma \curvearrowright X$, $\nu \in \mathrm{Prob}(X)$ be Γ-rigid, and let $\Gamma \curvearrowright (Z, m)$ be a p.m.p. action. Then there is no amenable von Neumann subalgebra $N \subseteq \Gamma \ltimes \big(B(L^2(X,\nu))\overline{\otimes}L^\infty(Z,m)\big)$ that contains $\Gamma \ltimes L^\infty(X \times Z, \nu \times m)$ as a proper subalgebra.
In particular, if $\Gamma \curvearrowright (X,\nu)$ is Zimmer amenable, then $\Gamma \ltimes L^\infty(X \times Z, \nu \times m)$ is maximal amenable in $\Gamma \ltimes \big(B(L^2(X,\nu))\overline{\otimes}L^\infty(Z,m)\big)$.

The above theorem generalizes and strengthens a recent result of Suzuki (2019).

Thanks!